Model for a Rigid, 3D Mechanism Inspired by Pop-Up Origami, and its Application to a Re-configurable, Physical Environment

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***Abstract*— Origami has had wide-ranging application in mechatronics, robotics, design, and aerospace engineering. This paper offers a model for a rigid, three-dimensional mechanism inspired by "pop-up" origami (i.e., kirigami). In pop-up origami, a cut is introduced to the folded sheet to expand formal possibilities. We present vertex and parallel pop-up origami mechanisms, model the former using the Denavit-Hartenberg Convention, and present a case study that we are actively developing that harnesses the capacity of origami to fold and unfold on demand. We explore this case, calculating its actuation forces, while recognizing that the model presented here has potential to generalize widely.**

I. INTRODUCTION

Origami is the ancient art of folding a single sheet of paper to create a three-dimensional sculpture. In recent years, origami has received attention from researchers in mechatronics and robotics as a potential for applications at very small physical scales. For example, [1] presents a sheet, 1.7 cm square, that self-folds into a functional 3D robot that can walk, swim, and then dissolve in liquid. At the other end of the physical scale, origami structures have served as the basis for habitable, physical environments. The Miura Ori pattern of origami has, for instance, been applied to form the structural envelope of a chapel building [2]. Origami has also served as the basis for a variety of mechanical systems [3], from nano-devices [4] to heart stents [5], solar panels [6], and mirrors [7] [8].

While origami is mostly recognized as a three-dimensional sculpture formed by folding a sheet of paper, a variation of origami called kirigami, otherwise known as "pop-up" origami, introduces cuts into the folded sheet of paper to expand the formal possibilities of the resulting form [9] [10] [11]. In this paper, we characterize and explore the potential of pop-up origami for mechanical systems of wide-ranging applications at wide-ranging physical scales.

One property of origami (including the pop-up variant) that benefits mechanical systems is its capacity to fold and unfold on demand. It is this property that we harness in our own design of a suite of physically re-configurable outdoor furniture to be installed in a public, urban square which, for this paper, serves as a case study.

A. *From Pop-Up Books to Mechanical Systems*

Paul Jackson's *Cut and Fold Techniques for Pop-Up Designs* [12] provides a comprehensive introduction to the art of creating pop-up (origami) books using folded paper. Jackson describes the most basic elements of any pop-up origami mechanism and the techniques for creating one using paper. Generalizations of the designs from Jackson's book provided the foundation and inspiration for the models produced in this paper.

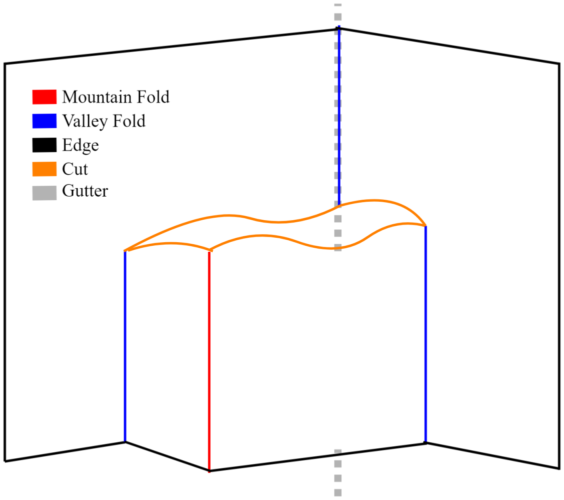

Fig. 1: Parallel pop-up mechanism in open configuration

We describe pop-up origami mechanisms using accepted terminology found in Jackson's book. These mechanisms are briefly explained here and visualized in Fig. 1. In order to be classified as a "pop-up," a paper mechanism must meet the following criteria:

- The mechanism is created from one paper sheet.
- The mechanism must have exactly four straight folds.
- The mechanism possesses two flat-folding configurations, "open flat" (Fig. 2) and "closed flat," such that a book could be fully opened or fully closed without violating the mechanism's range of motion.
- One or both of the center folds must be co-linear with the gutter.
- The mechanism must contain exactly one cut which is entirely on the interior of the paper.
- All folds begin at the cut and terminate at the edge of the paper.

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- The cut begins and ends at the start of the two outermost folds, and need not be a straight line.

These criteria allow for eight different fold locations, each with two different fold types, for a total of 1120 different folding patterns. Of these, only 8 patterns are physically realizable (i.e. it is not possible to create a mechanism where all folds are mountain type). Jackson elaborates these valid configurations in *Cut and Fold Techniques for Pop-Up Designs* [12].

B. The Engineering of Larger-Scale Pop-Up Mechanisms

When considering the applications of pop-up design at a larger physical scale (e.g. our case study), the mechanism must meet the following amended criteria. The amendments primarily recognize the engineering limitations of a large-scale pop-up such as replacing paper folds with mechanical joints.

- The mechanism is constructed of exactly four flat panels, attached by four revolute joints.
- The neighboring edges panels are straight and parallel to the shared joint's axis of rotation.
- The mechanism has at least one flat-folding configuration.

The main difference between paper and large-scale construction is the thickness of materials. At paper-scale, thickness is negligible and building materials are highly flexible, but large-scale construction introduces greater opportunity for collision as panel thickness increases, and materials are more rigid.

II. GENERAL FORMS OF POP-UP ORIGAMI MECHANISMS

Jackson's pop-ups fall into two broad categories, referred to in this paper as "vertex" and "parallel" type mechanisms, which Winder et al. describe as one-piece, single-slit planar and one-piece single-slit spherical mechanisms, respectively. These mechanisms each constitute one degree-of-freedom, four-bar mechanisms [9]. These two cases are characterized by the relationship between the folds of the mechanism, and allow for rigid, three-dimensional motion without buckling or locking.

A. Vertex Mechanism

The vertex mechanism type (Fig. 2) demonstrates spherical motion about a fixed point (vertex) in space. Instead of the parallel fold axes shown in Fig. 1, these axes converge at a point on the gutter. This has been shown to allow for motion by creating a spherical four-bar mechanism [9].

B. Parallel Mechanism

The parallel mechanism (Fig. 1) is a special case of the vertex mechanism where the vertex point is infinitely far from the intersection of the cut and the gutter. However, it is easier to express this as its own type since it behaves as a planar, four-bar mechanism.

III. GEOMETRIC MODELS OF THE VERTEX POP-UP MECHANISM

For this paper, the vertex pop-up mechanism is chosen to explore further, given that it has application to the authors' use case (which will be elaborated further in Section V).

In order to solve for the structural mechanics of a pop-up mechanism, it is necessary to locate points of force application in a common frame. This is accomplished by defining points relative to frames attached to each link and using a homogeneous linear transform to find those same points relative to other frames.

A. Denavit-Hartenberg Convention

The Denavit-Hartenberg (D-H) convention [13] is a convenient way to describe kinematic transformations between coordinate frames that are related by either revolute or prismatic joints. Although normally used in robotics applications for determining the kinematic chain of a multiple degree-of-freedom end manipulator [14], these conventions are a good tool for describing the configuration of each joint of a pop-up mechanism.

Craig provides instructions for calculating robotic kinematics using what has been called "modified" D-H parameters. The definition of each parameter is shown in Table I and considered in [14].

TABLE I: D-H parameter definitions

Parameter	**Definition**
a_i	Linear distance from z_i to z_{i+1} measured in the x_i direction
α_i	Angular displacement between z_i and z_{i+1} measured about the x_i axis
d_i	Linear distance from x_{i-1} to x_i measured in the z_i direction
θ_i	Angular displacement between x_{i-1} and x_i measured about the z_i axis

B. Angle Definitions

Fig. 2 shows the necessary parameters to fully define the joint axes of the pop-up mechanism. Only three independent vertex angles ($alpha$ in Table I) are required since the fourth is dependent on the other three. Fig. 3 shows the placement of the link frames for each link.

In order to specify a given pose of the mechanism, the angles of each revolute joint of the mechanism must be determined. In the case of a pop-up mechanism, specifying one joint angle (θ in Table I) determines all of the angles. For the vertex pop-up, these angles are governed by spherical trigonometry. Fig. 4 defines intermediate variables that will be used to express the joint angles of the mechanism in terms of the single input angle (ϕ in Fig. 4). Spherical trigonometry provides the following relationships between the internal angles [15].

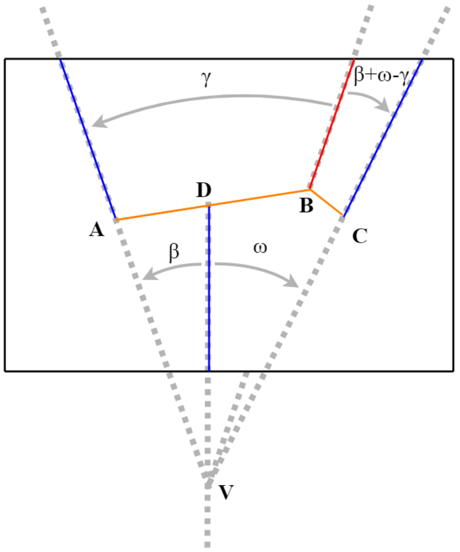

Fig. 2: Open flat configuration for vertex pop-up kinematics

$$\delta = atan\left(\frac{sin(\phi)}{cot(\gamma)sin(\beta) - cos(\beta)cos(\phi)}\right) \tag{1}$$

$$\epsilon = atan\left(\frac{sin(\phi)}{cot(\beta)sin(\gamma) - cos(\gamma)cos(\phi)}\right) \tag{2}$$

$$\psi = acos\Big(cos(\beta)cos(\gamma) + sin(\beta)sin(\gamma)cos(\phi)\Big) \tag{3}$$

$$\mu = acos\left(\frac{cos(\beta+\omega-\gamma) - cos(\omega)cos(\psi)}{sin(\omega)sin(\psi)}\right) \tag{4}$$

$$\kappa = acos\left(\frac{cos(\psi) - cos(\omega)cos(\beta+\omega-\gamma)}{sin(\omega)sin(\beta+\omega-\gamma)}\right) \tag{5}$$

$$\zeta = acos\left(\frac{cos(\omega) - cos(\psi)cos(\beta+\omega-\gamma)}{sin(\psi)sin(\beta+\omega-\gamma)}\right) \tag{6}$$

C. Link Frames

Given the link frames presented in Fig. 3 and the angles defined in Fig. 4, the D-H parameters defining the configuration of frame i are provided in Table I.

Using the D-H parameters, it is possible to describe points on each joint in different coordinate frames. The transformation between coordinate frames is accomplished using a homogeneous transform. A point whose coordinates are expressed in frame h (${}^{h}\vec{p}$) can be represented instead in frame k (${}^{k}\vec{p}$) using the relationship shown in 7. The construction of the homogeneous transformation matrix ($[{}^{k}_{h}T]$) using D-H parameters (see Table II) is well established and, consequently, will not be presented here [14].

$${}^{k}\vec{p} = [{}^{k}_{h}T]\,{}^{h}\vec{p} \tag{7}$$

TABLE II: transFORM Link Frame D-H Parameters

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$-\beta$	0	$\pi-\phi$
2	0	$-\gamma$	0	$\pi-(\epsilon+\zeta)$
3	0	$-(\beta+\omega-\gamma)$	0	$\pi-\kappa$
4	0	$-\omega$	0	$\pi-(\delta+\mu)$

IV. STRUCTURAL MECHANICS TO DETERMINE ACTUATION FORCE

The practical construction of a large scale pop-up mechanism is a main goal of this project. Because a pop-up possesses a single degree of freedom, the entire mechanism can be actuated with a single input force or torque. However, sizing an appropriate motor or other actuation device is not trivial when a mechanism exhibits complicated three-dimensional motion. Therefore, a simple analytic method for solving the actuation effort is derived.

A. Simplifying Assumptions

The spherical, four-bar mechanism formed by the vertex pop-up has one degree of freedom. However, in an arbitrary four-link, closed-chain mechanism where the joint axes are neither parallel nor convergent on fixed point, the mechanism is over-constrained. Therefore, a few simplifying assumptions must be made in order to solve for the actuation force using the second law of motion and the Newton-Euler equations. These simplifications are provided below as well as shown as a diagram in Fig. 5 and Figs. 6 – 8.

- Joints are modelled as friction-less. Therefore, each revolute joint has only two unknown reaction moments.
- Link 0 is assumed rigidly attached to ground.
- Link 3 is assumed effectively mass-less (or otherwise supported against the force of gravity).
- The joint at point C is modelled as a ball-and-socket. Therefore, all reaction moments at this joint are zero.
- The joint at point D is modelled as free-floating in the z-direction. That is, point D supplies no reaction force along the joint axis.

B. Static Solution

With the above simplifications, free body diagrams (Figs. 6 – 8) can be constructed. There are 17 unknowns: $F_{A,x}$, $F_{A,y}$, $F_{A,z}$, $F_{B,x}$, $F_{B,y}$, $F_{B,z}$, $F_{C,x}$, $F_{C,y}$, $F_{C,z}$, $F_{D,x}$, $F_{D,y}$, $M_{A,x}$, $M_{A,y}$, $M_{B,x}$, $M_{B,y}$, $M_{D,x}$, $M_{D,y}$. There are three Newtonian force balance equations each for links 1, 2, and 3 – for a total of nine force equations. There are three Newtonian moment balance equations each for links 1, 2, and 3 – for a total of nine moment equations. With 18 equations and 17 unknowns, a designer is free to choose an appropriate unknown actuation effort and solve for values to create static equilibrium.

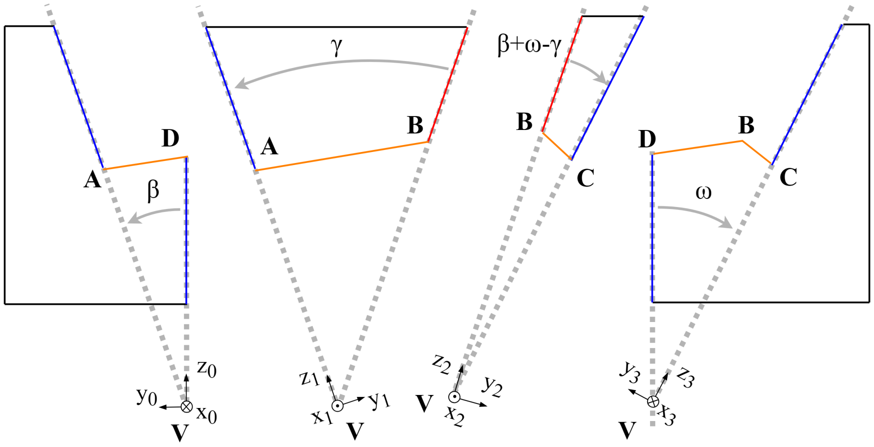

Fig. 3: Vertex pop-up link frame definitions

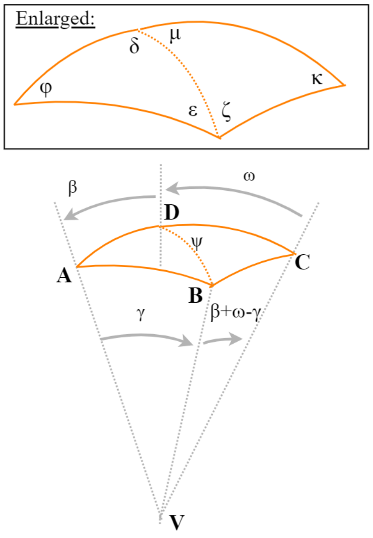

Fig. 4: Diagram for solving spherical joint angle relationships

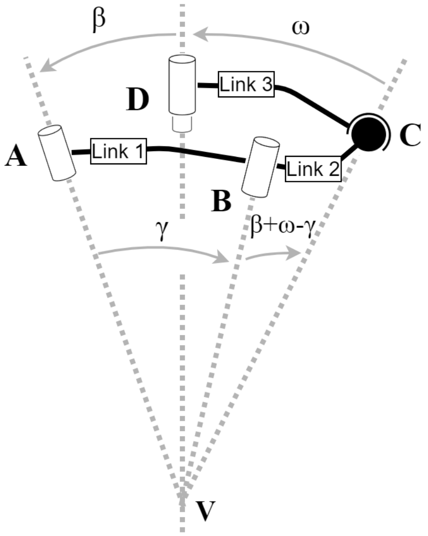

Fig. 5: Diagram of mechanical simplifications. Please note the joint type at A and B is revolute, while at D there is a combination revolute-sliding joint.

V. USE CASE: "TRANSFORM," A RECONFIGURABLE, CYBER-PHYSICAL ENVIRONMENT

Fig. 9 shows computer renderings of three configurations of what we call "transFORM." The project is motivated by the emergence of social networks and apps that have reduced the importance of physical space as a locus for social interaction and place attachment [16] [17] [18] [19]. As a means to recapture the importance of public outdoor spaces [20] [21], our research team proposes this reconfigurable, cyber-physical environment at room-scale, installed in a public, urban square. We strive to "create fresh urban relationships, processes, and patterns that have the social and cultural qualities we seek for the twenty-first century" [22].

In practical terms, transFORM is a series of hinged, responsive panels with embedded lighting, audio, and displays that transform according to needs. A pop-up origami mechanism permits the hinged, single sheet to reconfigure ("transform") into different utilitarian and evocative configurations that facilitate lounging, meeting, reading, working, exchanging, accessing information, and generating information. Fig. 9 shows three such configurations.

A. *Modelling Parameters*

The D-H parameters for the transFORM mechanism are provided in Table III. It is important to note that in this application, ω and β are equal. Furthermore, the link angles θ_i are left in variable form as they are dependent on 1 – 6. Because it is one degree-of-freedom, upon specifying ϕ, all other angles can be determined.

B. *Force Analysis Results*

The actuation effort solved for in this analysis is the reaction force necessary to accomplish static equilibrium, applied at the far corner of link 3 (see Fig. 11). For this analysis, centers of gravity are estimated using a computational geometry program (Solidworks, 2017). A MATLAB script solves

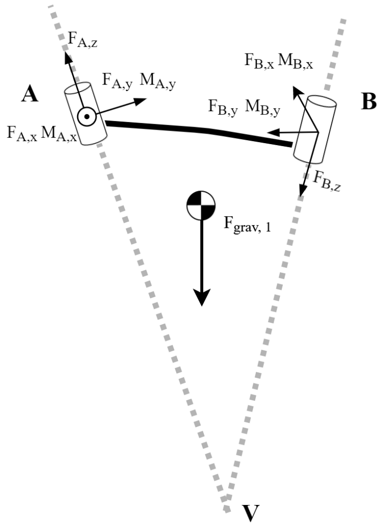

Fig. 6: Free body diagram for link 1.

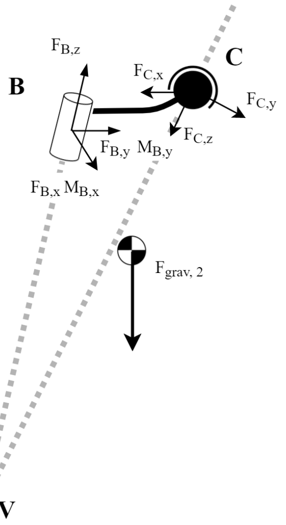

Fig. 7: Free body diagram for link 2.

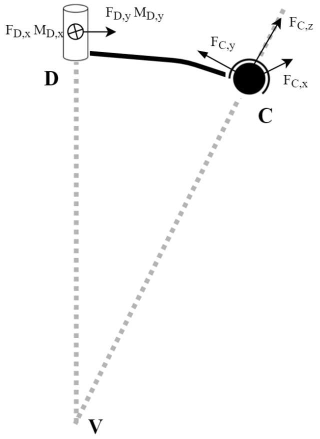

Fig. 8: Free body diagram for link 3.

TABLE III: Vertex Pop-up Link Frame D-H Parameters

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$-25°$	0	$\pi - \phi$
2	0	$-35°$	0	$\pi - (\epsilon + \zeta)$
3	0	$-15°$	0	$\pi - \kappa$
4	0	$-25°$	0	$\pi - (\delta + \mu)$

for the unknown force across the entire range of possible configurations, and the results are presented in Fig. 10. An actuator placed at the location shown in Fig. 11 would need to exert a maximum estimated reaction force of 200 Newtons or 45 lbs to keep the mechanism in equilibrium. Therefore, a motor capable of applying approximately 600 Newtons or 135 lbs of force is recommended for this mechanism.

VI. Further Applications

More broadly and beyond the transFORM project introduced here, the vertex pop-up mechanism offers a replicable and modular platform for future development. At large scale, we can - for instance - envision pop-up origami mechanisms at the core of flat-packed emergency housing and mobile hospital units to provide a variety of critical resources in response to natural or human-made disasters.

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Fig. 9: A case study of the vertex pop-up mechanism, transFORM (three configurations shown above) is 2.5m (approx. 8ft) tall in its flattened configuration (the right-most image, shown at night). The pop-up mechanism constitutes the bottom-most four panels.

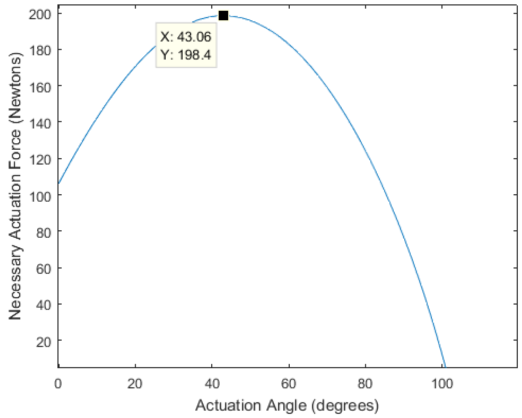

Fig. 10: transFORM actuation force solution results. Actuation angle is the defined in Fig. 11.

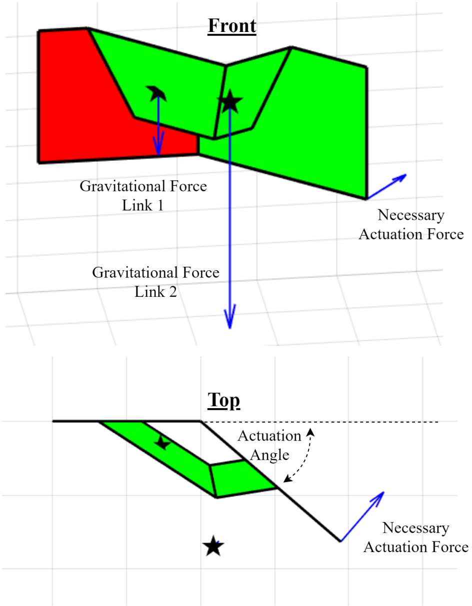

Fig. 11: transFORM diagram generated by MATLAB. Stars denote centers of mass of individual panels. The second center of mass "floating" in the air is due to fly-away panels attached to panel (not shown - see Fig. 9). Forces of interest shown in blue. Red panel (link 0) is rigidly attached to ground. Link 3 is modelled as mass-less.

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